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Uniform stability estimates for solutions and their gradients to the Boltzmann equation: A unified approach

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Abstract

As to the Cauchy problem for the spatially inhomogeneous Boltzmann equation with cut-off, we prove uniform stability estimates for solutions and their gradients in a unified and elementary way.

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1. Introduction

In the kinetic theory, the Boltzmann equation arises as a mathematical model for a rarefied gas that describes the statistical evolution of one-particle distribution function $f(x, v, t)$ having position $x \in \mathbb{R}^3$ and velocity $v \in \mathbb{R}^3$ at time $t \geqslant 0$. In the absence of external forces, the Cauchy problem for the inhomogeneous Boltzmann equation takes the form of

$$\text{(CB)} \qquad \begin{cases} \partial_t f + v \cdot \nabla_x f = Q(f, f) & \text{in } \mathbb{R}^3 \times \mathbb{R}^3 \times (0, \infty), \\ f(x, v, 0) = f_0(x, v) & \text{on } \mathbb{R}^3 \times \mathbb{R}^3, \end{cases}$$

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where f_0 is a given nonnegative function and Q is the collision operator defined as the bilinear form

$$Q(f,g)(v)=\iint_{\mathbb{R}^3\times\mathbb{S}^2} B(v-v_*,\sigma)\big[f(v')g\big(v'_*\big)-f(v)g(v_*)\big]\,d\sigma\,dv_* \tag{1}$$

in which we omit the variables x, t for simplicity. Here (v, v_*) and (v', v'_*) stand for the velocity variables of two gas particles before and after collision, respectively. Due to the conservation of momentum and kinetic energy, they are related by the collision law

$$v'=v-\big[(v-v_*)\cdot\sigma\big]\sigma,\qquad v'_*=v_*+\big[(v-v_*)\cdot\sigma\big]\sigma, \tag{2}$$

where $\sigma\in\mathbb{S}^2$ denotes the direction of $v'-v$. The collision kernel B is a nonnegative function of $|v-v_*|$ and the deviation angle θ determined by

$$\cos\theta=\left(\frac{v-v_*}{|v-v_*|}\right)\cdot\sigma. \tag{3}$$

In dealing with our problem described below, it will be advantageous to consider the mild form of Boltzmann equation. With the customary notation $\phi^\sharp(x,v,t)=\phi(x+tv,v,t)$ for any function on $\mathbb{R}^3\times\mathbb{R}^3\times[0,\infty)$, it is defined to be the integral equation

$$\text{(MB)}\quad f^\sharp(x,v,t)=f_0(x,v)+\int_0^t Q^\sharp(f,f)(x,v,s)\,ds.$$

A continuous function f is said to be a classical solution if it is nonnegative, continuously differentiable in (x,t) and satisfies (CB) for all (x,v,t). A measurable function f is said to be a mild solution if it is nonnegative and satisfies (MB) for almost every $(x,v)\in\mathbb{R}^3\times\mathbb{R}^3$ and for all $t\geqslant 0$. To put in another way, we may identify a mild solution f as a nonnegative fixed point of the operator $f^\sharp\mapsto f_0+(Jf)^\sharp$, where

$$(Jf)(x,v,t)=\int_0^t Q(f,f)(x,v,s)\,ds. \tag{4}$$

Evidently, a classical solution is a mild solution.

In this paper we are mainly concerned with L^1 stability properties of solutions to the Boltzmann equation. To be more specific, given two solutions f, g corresponding to the initial data f_0, g_0, respectively, we are interested in studying if L^1 distances

$$\big\|f(t)-g(t)\big\|_{L^1}=\iint_{\mathbb{R}^3\times\mathbb{R}^3}\big|f(x,v,t)-g(x,v,t)\big|\,dx\,dv$$

are *stable* with respect to the distance $\|f_0-g_0\|_{L^1}$ as time t evolves.

Regarding the space of solutions, it would be of wide interest to consider those solutions f having finite mass, energy and entropy for all time, that is,

$$\sup_{t\geqslant 0}\iint_{\mathbb{R}^3\times\mathbb{R}^3} f(x,v,t)\big(1+|x|^2+|v|^2+\log f(x,v,t)\big)\,dx\,dv<+\infty. \tag{5}$$

However, we shall only consider those solutions that belong to a specific class of functions described as follows. We consider two positive functions h, m on $[0,\infty)$ such that h is continuous, decreasing,

$$\int_0^\infty h(r)\big(1+r^2\big)\,dr<+\infty \quad \text{and} \quad \int_0^\infty m(r)r^2\,dr<+\infty. \tag{6}$$

To each pair (h,m) of such functions, we denote by $O(h,m)$ the class of all measurable functions f on $\mathbb{R}^3\times\mathbb{R}^3\times[0,\infty)$ satisfying

$$\big|f(x,v,t)\big|\leqslant 2h\big(|x-tv|\big)m\big(|v|\big) \tag{7}$$

and by $D(h,m)$ the class of all measurable functions f_0 on $\mathbb{R}^3\times\mathbb{R}^3$ satisfying

$$0\leqslant f_0(x,v)\leqslant h\big(|x|\big)m\big(|v|\big). \tag{8}$$

In the theory of existence, it is known that there are global solutions to (MB) in the class $O(h,m)$ when the initial data belong to $D(h,m)$ if certain *smallness* condition is fulfilled (see e.g. [2,3,7,9]).

In this framework, most of existing L^1 stability results are focused on establishing uniform estimates of type

$$\sup_{t\geqslant 0}\big\|f(t)-g(t)\big\|_{L^1}\leqslant C\|f_0-g_0\|_{L^1} \tag{9}$$

for solutions $f, g\in O(h,m)$ corresponding to $f_0, g_0\in D(h,m)$ with some concrete choice of h, m, provided that

(i) the collision kernel B satisfies an angular cut-off assumption so that it is integrable over $\mathbb{S}^2$ (it is in general non-integrable due to singularity at $\theta=\pi/2$), and
(ii) the data are sufficiently small in certain sense.

For the details, we refer to the recent papers [4–6], which are directly related with the present work, and [1,8] for further references. As for the methods, the usual approach is to find a suitable differential inequality involving

$$\frac{d}{dt}\big\|f(t)-g(t)\big\|_{L^1} \quad \text{and} \quad \big\|f(t)-g(t)\big\|_{L^1}$$

and to exploit a refined version of Gronwall's lemma. In doing so, however, one needs to construct case by case an appropriate functional that controls $\|f(t)-g(t)\|_{L^1}$ from below and above, which causes a great deal of complexity.

Our point of view on stability is that it is an intrinsic property of the operator J and hence it can be obtained from a functional approach, that is, from mapping properties of J. Based on this point of view, we aim at establishing uniform L^1 stability estimates for solutions to the Boltzmann equation in a unified and elementary way. The assumption on the collision kernel B that we shall consider is the following form taken from Villani [11]:

$$0 \leqslant B(v-v_*,\sigma) \leqslant \Phi\big(|v-v_*|\big)b(\cos\theta). \tag{10}$$

(A1) The kinetic part Φ is a nonnegative measurable function on $[0,\infty)$ with

$$\Lambda(m,\Phi) = \sup_{v\in\mathbb{R}^3}\left[\int_{\mathbb{R}^3} m\big(|v_*|\big)\frac{\Phi(|v-v_*|)}{|v-v_*|}\,dv_*\right] < +\infty, \tag{11}$$

where m is a function described as in the definition of $O(h,m)$.

(A2) The angular part $b(\cos\theta)$ is supported in the set $0\leqslant\theta\leqslant\pi/2$ and satisfies Grad's cut-off hypothesis

$$\|b\|_{L^1(\mathbb{S}^2)} \equiv \int_{\mathbb{S}^2} b(\mathbf{k}\cdot\sigma)\,d\sigma = 2\pi\int_0^{\pi/2} b(\cos\theta)\sin\theta\,d\theta < +\infty \tag{12}$$

valid for any fixed unit vector $\mathbf{k}$ with $\mathbf{k}\cdot\sigma=\cos\theta$.

One of our principal results is the following L^1 stability.

Theorem 1. *Under assumption* (10) *on B with* (A1), (A2), *put*

$$\lambda = \lambda(h,m,B) = 16\|h\|_1\|b\|_{L^1(\mathbb{S}^2)}\Lambda(m,\Phi). \tag{13}$$

Suppose that $f,g\in O(h,m)$ are mild solutions to the Boltzmann equation corresponding to the initial data $f_0,g_0\in D(h,m)$, respectively. If h,m are chosen so as to $\lambda<1$, then

$$\sup_{t\geqslant 0}\big\|f(t)-g(t)\big\|_{L^1} \leqslant \left(\frac{1}{1-\lambda}\right)\|f_0-g_0\|_{L^1}. \tag{14}$$

The key idea of our proof is to show that J is a Lipschitz mapping, with the Lipschitz constant λ, on the class $O(h,m)$ with respect to the metric

$$d_{\max}(f,g) = \iint_{\mathbb{R}^3\times\mathbb{R}^3}\Big(\sup_{t\geqslant 0}\big|(f-g)^\sharp(x,v,t)\big|\Big)\,dx\,dv. \tag{15}$$

We remark that the mapping properties of J are well understood with respect to the metric $\|f-g\|_{h,m}$ defined as in (22) below.

Adopting the same techniques and ideas, it is straightforward to obtain uniform gradient stability and BV-type estimate for classical solutions.

Theorem 2. *Under the same setting as in Theorem* 1*, suppose that $f, g \in O(h,m)$ are classical solutions to the Boltzmann equation corresponding to the initial data $f_0, g_0 \in D(h,m)$, respectively, and $\nabla_x f, \nabla_x g \in O(h,m)$. If h, m are chosen so as to $\lambda < 1/2$, then*

$$\sup_{t \geqslant 0} \bigl\| \nabla_x f(t) - \nabla_x g(t) \bigr\|_{L^1} \leqslant \left(\frac{1}{1-2\lambda} \right) \| \nabla_x f_0 - \nabla_x g_0 \|_{L^1}. \tag{16}$$

In particular, the following uniform BV-type estimate holds:

$$\sup_{t \geqslant 0} \bigl\| \nabla_x f(t) \bigr\|_{L^1} \leqslant \left(\frac{1}{1-2\lambda} \right) \| \nabla_x f_0 \|_{L^1}. \tag{17}$$

As we shall see below, our results are applicable in the special case when $\Phi(|z|) = |z|^\gamma$ $(-2 < \gamma \leqslant 1)$ and $h(r) = m(r) = (1+r^2)^{-\beta/2}$ $(\beta > 3)$, for instance, which improves the aforementioned stability results greatly.

2. Proofs of main theorems

As it is standard, the cut-off conditions (10), (12) on B enables us to decompose the collision operator Q into $Q(f,g) = Q_+(f,g) - Q_-(f,g)$ in which the loss term Q_- is given by $Q_-(f,g) = f(Lg)$ with

$$(Lg)(x,v,t) = \iint\limits_{\mathbb{R}^3 \times \mathbb{S}^2} B(v - v_*, \sigma) f(x, v_*, t)\, d\sigma\, dv_*. \tag{18}$$

Given any measurable function ϕ on $\mathbb{R}^3 \times \mathbb{R}^3 \times [0,\infty)$, we shall write

$$\bigl\| \phi(t) \bigr\|_{L^1} = \iint\limits_{\mathbb{R}^3 \times \mathbb{R}^3} \bigl| \phi(x,v,t) \bigr|\, dx\, dv \quad (t \geqslant 0).$$

Lemma 2.1. *Let f, g be nonnegative and measurable. Then*

(a) $\| Q_+^\sharp(f,g)(t) \|_{L^1} = \| Q_-^\sharp(f,g)(t) \|_{L^1}$ $(t > 0)$*, and*
(b) $\| Q_-^\sharp(f,g)(t) \|_{L^1} = \| Q_-^\sharp(g,f)(t) \|_{L^1}$ $(t > 0)$.

Proof. To prove (a), we use Fubini's theorem to write $\| Q_+^\sharp(f,g)(t) \|_{L^1}$ as

$$\iiiint\limits_{\mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{S}^2} B f^\sharp\bigl(x + t(v - v'), v', t\bigr) g^\sharp\bigl(x + t\bigl(v - v'_*\bigr), v'_*, t\bigr)\, d\sigma\, dx\, dv_*\, dv,$$

where $B = B(|v - v_*|, \sigma)$. Changing variables $x + t(v - v') \to x$ and then interchanging the order of integrations, we see that it equals to

$$\iint_{\mathbb{R}^3\times\mathbb{S}^2} \left[\iint_{\mathbb{R}^3\times\mathbb{R}^3} Bf^\sharp(x, v', t) g^\sharp\big(x + t\big(v' - v'_*\big), v'_*, t\big)\, dv\, dv_* \right] d\sigma\, dx. \tag{19}$$

For each fixed $\sigma \in \mathbb{S}^2$, the collision transformation (2) satisfies

$$|v - v_*| = \big|v' - v'_*\big| \quad \text{and} \quad \det\left[\frac{\partial(v', v'_*)}{\partial(v, v_*)}\right] = 1.$$

It follows from changing variables $(v, v_*) \to (v', v'_*)$ that (19) equals to

$$\begin{aligned}
&\iiiint_{\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{S}^2} Bf^\sharp(x, v, t) g^\sharp\big(x + t(v - v_*), v_*, t\big)\, d\sigma\, dv_*\, dx\, dv \\
&\quad = \iint_{\mathbb{R}^3\times\mathbb{R}^3} f^\sharp(x, v, t)(Lg)^\sharp(x, v, t)\, dx\, dv = \big\| Q_-^\sharp(f, g)(t)\big\|_{L^1}.
\end{aligned} \tag{20}$$

To prove (b), simply change variables $x + t(v - v_*) \to x$ in (20). □

We now prove a bilinear estimate for the collision operator Q that will play a crucial role in deriving our results. Let X denote the space of measurable functions f on $\mathbb{R}^3 \times \mathbb{R}^3 \times [0, \infty)$ such that $M^\sharp(f) \in L^1(\mathbb{R}^3 \times \mathbb{R}^3)$, where $M^\sharp(f)$ stands for the maximal function of f along the characteristic lines defined by

$$M^\sharp(f)(x, v) = \sup_{t\geqslant 0}\big| f(x + tv, v, t)\big| = \sup_{t\geqslant 0}\big| f^\sharp(x, v, t)\big|. \tag{21}$$

We identify $O(h, m)$ as the closed ball $O(h, m) = \{\|f\|_{h,m} \leqslant 2\}$ where

$$\|f\|_{h,m} = \sup_{(x,v)\in\mathbb{R}^3\times\mathbb{R}^3} \big|M^\sharp(f)(x, v)\big| / \big[h\big(|x|\big)m\big(|v|\big)\big]. \tag{22}$$

Lemma 2.2. *Suppose that the collision kernel B satisfies assumption* (10) *with* (A1), (A2). *Then the collision operator Q maps $X \times O(h, m)$ boundedly into $L^1(\mathbb{R}^3 \times \mathbb{R}^3 \times [0, \infty))$ with*

$$\int_0^\infty \big\| Q(f, g)(t)\big\|_{L^1}\, dt \leqslant \big(4\|h\|_1 \|b\|_{L^1(\mathbb{S}^2)} \Lambda(m, \Phi)\big) \big\| M^\sharp(f)\big\|_{L^1} \|g\|_{h,m} \tag{23}$$

for all $f \in X$ and $g \in O(h, m)$.

Proof. By (a) of Lemma 2.1, we have

$$\begin{aligned}\left\|Q(f,g)(t)\right\|_{L^1} &= \left\|Q^\sharp(f,g)(t)\right\|_{L^1} \leqslant 2\left\|Q^\sharp_-(f,g)(t)\right\|_{L^1}\\ &\leqslant 2\iint\limits_{\mathbb{R}^3\times\mathbb{R}^3}\left|f^\sharp(x,v,t)\right|\left|(Lg)^\sharp(x,v,t)\right|dx\,dv.\end{aligned}$$

Integrating both sides with respect to dt, we obtain

$$\int\limits_0^\infty \left\|Q(f,g)(t)\right\|_{L^1}dt \leqslant 2\iint\limits_{\mathbb{R}^3\times\mathbb{R}^3} M^\sharp(f)(x,v)\left[\int\limits_0^\infty \left|(Lg)^\sharp(x,v,t)\right|dt\right]dx\,dv.$$

The integral inside the bracket is easily seen to be bounded by

$$\|g\|_{h,m}\|b\|_{L^1(\mathbb{S}^2)}\int\limits_{\mathbb{R}^3}\Phi\big(|v-v_*|\big)m\big(|v_*|\big)\left[\int\limits_0^\infty h\big(\left|x+t(v-v_*)\right|\big)\,dt\right]dv_*. \tag{24}$$

Since h is decreasing, we have

$$\int\limits_0^T h\big(\left|x+t(v-v_*)\right|\big)\,dt \leqslant 2\int\limits_0^{T/2} h\big(t|v-v_*|\big)\,dt \leqslant \frac{2\|h\|_1}{|v-v_*|} \tag{25}$$

uniformly in $x\in\mathbb{R}^3$ and $T>0$ (see e.g. [2]). Thus (24) is bounded by

$$2\|g\|_{h,m}\|h\|_1\|b\|_{L^1(\mathbb{S}^2)}\int\limits_{\mathbb{R}^3} m\big(|v_*|\big)\frac{\Phi(|v-v_*|)}{|v-v_*|}\,dv_*,$$

which yields from the definition of $\Lambda(m,\Phi)$ the estimate (23) right away. □

With the aid of these two lemmas, we now prove that the operator J, defined as in (4), is a Lipschitz mapping on the space $O(h,m)$ with respect to the metric $d_{\max}$ defined as in (15).

Lemma 2.3. *Suppose that the collision kernel B satisfies assumption* (10) *with* (A1)*,* (A2)*. Then the operator J is Lipschitz on $O(h,m)$ with*

$$d_{\max}(Jf,Jg) \leqslant \lambda d_{\max}(f,g) \quad \textit{for all } f,g\in O(h,m). \tag{26}$$

Proof. From the definition of J, it is clear that

$$M^\sharp(Jf-Jg)(x,v) \leqslant \int\limits_0^\infty \left|Q^\sharp(f,f)(x,v,t)-Q^\sharp(g,g)(x,v,t)\right|dt.$$

In view of the bilinearity of Q, we have

$$\begin{aligned}Q^\sharp(f,f)-Q^\sharp(g,g)&=\frac{1}{2}\big[Q^\sharp(f-g,f+g)+Q^\sharp(f+g,f-g)\big]\\&=Q^\sharp\left(f-g,\frac{f+g}{2}\right)+Q^\sharp\left(\frac{f+g}{2},f-g\right).\end{aligned}$$

It follows from the symmetry property (b) of Lemma 2.1 that

$$d_{\max}(Jf,Jg)\leqslant 2\int_0^\infty\left\|Q^\sharp\left(f-g,\frac{f+g}{2}\right)(t)\right\|_{L^1}dt. \tag{27}$$

Since $(f+g)/2\in O(h,m)$ and $f-g\in X$, it follows from an application of Lemma 2.2 that the right side of (27) is bounded by

$$16\|h\|_1\|b\|_{L^1(\mathbb{S}^2)}\Lambda(m,\Phi)\left\|M^\sharp(f-g)\right\|_{L^1}=\lambda d_{\max}(f,g),$$

which implies the desired estimate. □

We are now ready to prove our main results.

Proof of Theorem 1. As f, g are mild solutions, it follows from the definition of J that

$$(f-g)^\sharp(x,v,t)=(f_0-g_0)(x,v)+(Jf-Jg)^\sharp(x,v,t),$$

which yields the pointwise inequality

$$M^\sharp(f-g)(x,v)\leqslant|f_0-g_0|(x,v)+M^\sharp(Jf-Jg)(x,v).$$

Integrating both sides with respect to $dx\,dv$, it follows from Lemma 2.3 that

$$\begin{aligned}d_{\max}(f,g)&\leqslant\|f_0-g_0\|_{L^1}+d_{\max}(Jf,Jg)\\&\leqslant\|f_0-g_0\|_{L^1}+\lambda d_{\max}(f,g),\end{aligned}$$

which implies the estimate

$$d_{\max}(f,g)\leqslant\frac{1}{1-\lambda}\|f_0-g_0\|_{L^1}. \tag{28}$$

Since it is evident that

$$\sup_{t\geqslant0}\left\|f(t)-g(t)\right\|_{L^1}=\sup_{t\geqslant0}\left\|(f-g)^\sharp(t)\right\|_{L^1}\leqslant\left\|M^\sharp(f-g)\right\|_{L^1}=d_{\max}(f,g),$$

the estimate (28) gives the desired inequality (14). □

Proof of Theorem 2. Differentiating under the integral sign, we have

$$(\nabla_x f)^\sharp(x,v,t) = (\nabla_x f_0)(x,v) + \int_0^t Q^\sharp(\nabla_x f, f)(x,v,s)\,ds + \int_0^t Q^\sharp(f, \nabla_x f)(x,v,s)\,ds \tag{29}$$

for a classical solution f satisfying the assumption of Theorem 2. Considering this identity for g and then proceeding as in the proof of Theorem 1, it is plain to verify the estimate (16). □

3. Special models

In this section we consider some familiar models of collision operator and discuss which functions h, m are admissible in our theorems. For the precise meaning and discussions on these models, we refer to the review article [11].

1. (*Hard-Spheres Model*) $B(v - v_*, \sigma) = |v - v_*|\cos\theta$.
   In this case, we have

$$\Lambda(m, \Phi) \leqslant \big\| m(|v|)\big\|_{L^1(\mathbb{R}^3)} = 4\pi \int_0^\infty m(r) r^2\,dr$$

and Theorems 1 and 2 are applicable with any h, m described as in (6).

2. (*Power-Potentials Models*) $\Phi(|z|) = |z|^\gamma$ $(-2 < \gamma \leqslant 1)$.
   In this case, we have

$$\Lambda(m, B) = \sup_{v\in\mathbb{R}^3} \int_{\mathbb{R}^3} |v - v_*|^{-3+(\gamma+2)} m\big(|v_*|\big)\,dv_*$$

in which the integral may be interpreted as the Riesz potential of order $\gamma + 2$ of the radial function $m(|v_*|)$. In general, Riesz potential operators behave badly for L^1 functions. However, if m is bounded, then a trivial estimate shows that

$$\Lambda(m, B) \leqslant 4\pi \left\{ \frac{\|m\|_\infty}{\gamma + 2} + \int_0^\infty m(r) r^2\,dr \right\}.$$

Thus Theorems 1 and 2 are applicable with any h, m described as in (6) if m is bounded. To list some of interesting cases, we have

$$\text{(i)}\quad m(r) = \exp(-\alpha r) \quad\text{or}\quad \exp\big(-\alpha r^2\big) \quad (\alpha > 0),$$

$$\text{(ii)}\quad m(r) = \big(1 + r^2\big)^{-\beta/2} \quad (\beta > 3),$$

$$\text{(iii)}\quad m(r) = \frac{[\log(1+r)]^\alpha}{(1+r^2)^{\beta/2}} \quad (\alpha \geqslant 0,\ \beta > 3).$$

Here we may take h to be any function defined in this list.

4. Global existence results

Our stability estimates are based on the hypotheses that mild solutions exist globally and lie in the class $O(h,m)$. With more restrictive h, m and more stringent *smallness* condition, it is in fact possible to develop a theory of global existence that suits to these hypotheses. Although the subject is quite classical, we shall present here two global existence theorems for the sake of completeness.

Given a positive measurable function m on $[0,\infty)$, we put

$$\Omega(m,B)=\sup_{v\in\mathbb{R}^3}\iint_{\mathbb{R}^3\times\mathbb{S}^2}\frac{\Phi(|v-v_*|)b(\cos\theta)m(|v'|)m(|v_*'|)}{|v-v_*|\cos\theta\sin\theta m(|v|)}\,d\sigma\,dv_*. \tag{30}$$

In accordance with [2], we denote by $\mathcal{H}$ the class

$$\mathcal{H}=\Big\{h\in C\big([0,\infty)\big)\cap L^1\big([0,\infty)\big):\ h\geqslant 0,\ h \text{ is decreasing and}$$
$$\sup_{r>0}h(r/\sqrt{2})/h(r)=h_c<+\infty\Big\}.$$

Theorem 4.1. *For a positive measurable function m on $[0,\infty)$ and $h\in\mathcal{H}$, suppose that B satisfies condition* (10) *with* (A1), (A2) *and*

$$\nu=8\|h\|_1\big[h_c\Omega(m,B)+2\|b\|_{L^1(\mathbb{S}^2)}\Lambda(m,\Phi)\big]<1. \tag{31}$$

Then the Cauchy problem (CB) *for the Boltzmann equation with an initial data $f_0\in D(h,m)$ has a unique mild solution $f\in O(h,m)$.*

Outline of proof. Fix $f_0\in D(h,m)$ and consider

$$(Kf)^\sharp(x,v,t)=f_0(x,v)+\int_0^t Q^\sharp(f,f)(x,v,s)\,ds.$$

We recall that the space $O(h,m)$ is complete with respect to the metric $d_{h,m}(f,g)=\|f-g\|_{h,m}$. Let us put

$$\nu_+=8\|h\|_1h_c\Omega(m,B),\qquad \nu_-=8\|h\|_1\|b\|_{L^1(\mathbb{S}^2)}\Lambda(m,\Phi)$$

so that $\nu=\nu_++2\nu_-$. The scheme of proof consists of two steps.

Step 1. The operator K maps $O(h,m)$ into itself and is a contraction. Thus K has a unique fixed point $f\in O(h,m)$.

An inspection shows that it is a simple consequence of the estimates

$$\int_0^t\big|Q^\sharp(f,f)(x,v,s)\big|\,ds\leqslant(\nu_++\nu_-)h\big(|x|\big)m\big(|v|\big),$$
$$\|Kf-Kg\|_{h,m}\leqslant(\nu_++\nu_-)\,\|f-g\|_{h,m} \tag{32}$$

valid for $f, g \in O(h, m)$, which in turn follow from the bilinear estimates

$$\int_0^t \left|Q_+^\sharp(f, g)(x, v, s)\right| ds \leqslant \frac{\nu_+}{4} \|f\|_{h,m} \|g\|_{h,m} h\big(|x|\big) m\big(|v|\big),$$

$$\int_0^t \left|Q_-^\sharp(f, g)(x, v, s)\right| ds \leqslant \frac{\nu_-}{4} \|f\|_{h,m} \|g\|_{h,m} h\big(|x|\big) m\big(|v|\big). \tag{33}$$

While the estimate for the loss term in (33) is straightforward, the estimate for the gain term in (33) is subtle. In fact, one needs to observe that

$$\int_0^t h\big(\left|x + s(v - v')\right|\big) h\big(\left|x + s\big(v - v'_*\big)\right|\big) ds$$
$$\leqslant 2h_c h\big(|x|\big) \int_0^{t/2} h\big(s \min\big(|v - v'|, \left|v - v'_*\right|\big)\big) ds \leqslant \frac{2h_c \|h\|_1 h(|x|)}{|v - v_*| \cos\theta \sin\theta}, \tag{34}$$

where the first inequality results from $h \in \mathcal{H}$ (see [2]) and the second results from the collision laws.

Step 2. The unique fixed point of K is nonnegative for all $t \geqslant 0$.

In [7], Kaniel and Shinbrot invented a remarkable method of settling down this question of nonnegativity. Fixing the variables x, v, let

$$\zeta_0(t) = 0, \qquad \phi_0^\sharp(t) = 2h\big(|x|\big) m\big(|v|\big).$$

For $k \geqslant 1$, we define $\zeta_k(t), \phi_k(t)$ recursively as the solutions to

$$\begin{cases} \dfrac{d}{dt}\zeta_k^\sharp(t) + \zeta_k^\sharp(t)(L\phi_{k-1})^\sharp(t) = Q_+^\sharp(\zeta_{k-1}, \zeta_{k-1})(t), \\ \dfrac{d}{dt}\phi_k^\sharp(t) + \phi_k^\sharp(t)(L\zeta_{k-1})^\sharp(t) = Q_+^\sharp(\phi_{k-1}, \phi_{k-1})(t), \\ \zeta_k(0) = \phi_k(0) = f_0. \end{cases} \tag{35}$$

As the explicit representations are available, it is not hard to observe that $(\zeta_k), (\phi_k)$ lie in the space $O(h, m)$ and

$$0 \leqslant \zeta_1(t) \leqslant \zeta_2(t) \leqslant \cdots \leqslant \phi_2(t) \leqslant \phi_1(t) \leqslant \phi_0(t).$$

If we let $\zeta(t) = \lim \zeta_k(t)$, $\phi(t) = \lim \phi_k(t)$, then we have

$$0 \leqslant \zeta^\sharp(t) \leqslant \phi^\sharp(t) \leqslant 2h\big(|x|\big) m\big(|v|\big). \tag{36}$$

Applying Lebesgue's dominated convergence theorem, it can be shown

$$\begin{cases} \zeta^\sharp(t) = (K\zeta)^\sharp(t) + \displaystyle\int_0^t Q_-^\sharp(\zeta, \zeta - \phi)(s)\,ds, \\ \phi^\sharp(t) = (K\phi)^\sharp(t) - \displaystyle\int_0^t Q_-^\sharp(\phi, \zeta - \phi)(s)\,ds. \end{cases} \tag{37}$$

It follows that

$$\begin{aligned} \|\zeta - \phi\|_{h,m} &\leqslant \|K\zeta - K\phi\|_{h,m} + \frac{\nu_-}{4}\|\zeta - \phi\|_{h,m}\|\zeta + \phi\|_{h,m} \\ &= (\nu_+ + 2\nu_-)\|\zeta - \phi\|_{h,m} = \nu\|\zeta - \phi\|_{h,m}. \end{aligned} \tag{38}$$

This estimate shows that $\zeta = \phi$ if $\nu < 1$. Eq. (37) then implies that $\zeta = \phi$ is a fixed point of K in the space $O(h, m)$. By the uniqueness, we conclude that $\zeta = \rho = f$, where f is the fixed point of K obtained in Step 1. Due to the property (36), hence, f is nonnegative for all time t. $\square$

Remark 4.1. In practice, it is often difficult to estimate $\Omega(m, B)$. For instance, refer to the paper [10] of Toscani who made use of Carlemen's identity to estimate $\Omega(m, B)$ in the special case when

$$\Phi(|z|) = |z|^\gamma, \quad b(\cos\theta) = \cos\theta \quad \text{and} \quad m(r) = \left(1 + r^2\right)^{-\beta/2} \quad (\beta > 3).$$

A typical element of $\mathcal{H}$ is $h(r) = (1 + r^2)^{-\beta/2}$ $(\beta > 0)$. However, it does not contain the family $h(r) = \exp(-\sigma r^2)$ $(\sigma > 0)$. In this case, there is a theorem analogous to Theorem 4.1. To state, we put

$$\widetilde{\Omega}(m, B) = \sup_{v \in \mathbb{R}^3} \iint_{\mathbb{R}^3 \times \mathbb{S}^2} \frac{\Phi(|v - v_*|)b(\cos\theta)m(|v'|)m(|v'_*|)}{|v - v_*|m(|v|)}\,d\sigma\,dv_*. \tag{39}$$

Theorem 4.2. *For a positive measurable function m on* $[0, \infty)$ *and* $h(r) = \exp(-\sigma r^2)$ $(\sigma > 0)$, *suppose that B satisfies* (10) *with* (A1), (A2) *and*

$$\mu = 4\sqrt{\frac{\pi}{\sigma}}\left[\widetilde{\Omega}(m, B) + 8\|b\|_{L^1(\mathbb{S}^2)}\Lambda(m, \Phi)\right] < 1. \tag{40}$$

Then the Cauchy problem (CB) *for the Boltzmann equation with an initial data $f_0 \in D(h, m)$ has a unique mild solution $f \in O(h, m)$.*

The proof is basically the same as that of Theorem 4.1. The only points that need modification are the value $\|h\|_1 = 2\sqrt{\pi/\sigma}$ and the estimate

$$\int_0^t h\big(\big|x+s(v-v')\big|\big)h\big(\big|x+s\big(v-v'_*\big)\big|\big)\,ds \leqslant \sqrt{\frac{\pi}{\sigma}}\frac{h(|x|)}{|v-v_*|} \tag{41}$$

in place of (34) (see [2] for the details).

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